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# Weak gauge principle and electric charge quantization

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## Abstract

Starting from a weak gauge principle we give a new and critical revision of the argument leading to charge quantization on arbitrary spacetimes. The main differences of our approach with respect to previous works appear on spacetimes with non-trivial torsion elements on its second integral cohomology group. We show that in these spacetimes there can be topologically non-trivial configurations of charged fields which do not imply charge quantization. However, the existence of a non-exact electromagnetic field always implies the quantization of charges. Another consequence of the theory for spacetimes with torsion is the fact that it gives rise to two natural quantization units that could be identified with the electric quantization unit (realized inside the quarks) and with the electron charge. In this framework the colour charge can have a topological origin, with the number of colours being related to the order of the torsion subgroup. Finally, we discuss the possibility that the quantization of charge may be due to a weak non-exact component of the electromagnetic field extended over cosmological scales.

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## 1. Introduction

The idea of justifying electric charge quantization by means of topological arguments goes back to Dirac [14]. He showed that the existence of a magnetic monopole would imply that both the charge of the monopole and the electric charge of any other particle in the Universe are quantized. Since the electromagnetic field diverges at the worldline of the magnetic monopole, the problem was essentially that of a charged particle moving on a spacetime with non-trivial topology.

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Wu and Yang presented in [40] a geometrical explanation for the charge quantization argument proposed by Dirac. These authors showed that the quantization of the charge of a magnetic monopole is completely equivalent to saying that the monopole can be described as a connection on a non-trivial principal  $U(1)$ -bundle over spacetime. Although the approach followed by these authors is suitable for the particular case of the magnetic monopole, as we shall see in due course their point of view is not general enough for studying charge quantization on arbitrary spacetimes.

Therefore, the purpose of our paper is to analyse, starting from basic and well-established physical principles, the most general conditions which lead to quantization of electric charge on spacetimes with non-trivial topology. According to recent research these spaces cannot be directly ruled out if we take into account our present cosmological knowledge [27–29].

We give a new and critical revision of the argument leading to charge quantization based on what we call the ‘weak gauge principle’ and without assuming *a priori* an identification between electromagnetism and principal  $U(1)$ -bundles with connections. The main difference of our approach with respect to the standard ones lies on the fact that the weak gauge principle only appears in the presence of (at least) two different charged matter fields in interaction with a gauge field. Note that one of them should be considered as the charged reference field allowing for a measure of the relative interference Aharonov–Bohm class with respect to the second charged field.

As we shall see, this extension of the gauge principle agrees with the ordinary gauge principle (referred to in the paper as the ‘strong gauge principle’) and makes no difference for spacetimes with trivial topology. In this class of spaces leads to the same results already present in the literature. However, the weak gauge principle has non-trivial physical implications on spacetimes with non-trivial second integer cohomology group with non-vanishing torsion. Note that this is a radical difference with previous works on this matter which have been based on manifolds without torsion.

Now let us briefly explain some of the arguments that have led us to introduce the weak gauge principle. On Minkowski spacetime, a charged matter field  $\psi$  with charge  $q_\psi$  changes under a gauge transformation  $A' = A + d\alpha$  as  $\psi' = e^{iq_\psi\alpha}\psi$ . If we consider another charged matter field  $\phi$  with charge  $q_\phi$ , then under the same gauge transform it changes as  $\phi' = e^{iq_\phi\alpha}\phi$  with  $d\alpha = d\bar{\alpha}$ . Since the spacetime is contractible we have  $\bar{\alpha} = \alpha + \text{const}$  and the difference in the gauge transformation of the two fields is a global phase which can be gauged away after a redefinition of  $\phi$ . Therefore, in this spacetime the usual formulation of the  $U(1)$  gauge invariance of the theory for two different charged matter fields is simply written as

$$A' = A + d\alpha, \quad \psi' = e^{iq_\psi\alpha}\psi, \quad \phi' = e^{iq_\phi\alpha}\phi,$$

with the same function  $\alpha$  for both fields (the strong gauge principle).

On the other hand, it is well known that on a general spacetime  $M$  the gauge field  $\{A^i\}_{i \in I}$  and the charged matter fields  $\{\phi^i\}_{i \in I}$ ,  $\{\psi^i\}_{i \in I}$  are described by families parametrized by a good covering  $\{U_i\}_{i \in I}$  of  $M$  (see section 3 for more details). Therefore the transformation of the fields under a gauge transformation on  $U_{ij} = U_i \cap U_j$  between  $U_j$  and  $U_i$  should be written as

$$A^i = A^j + d\beta^{ij}, \quad \psi^i = e^{iq_\psi\beta^{ij}}\psi^j, \quad \phi^i = e^{iq_\phi\bar{\beta}^{ij}}\phi^j.$$

Although for each contractible open set  $U_{ij}$  we still have  $\beta^{ij} = \bar{\beta}^{ij} + \text{const}$ , notice that in general, a non-trivial topology of the spacetime might make it impossible to globally gauge away the difference between the constants  $\{\beta^{ij}\}_{i \in I}$ ,  $\{\bar{\beta}^{ij}\}_{i \in I}$ . Indeed a  $U(1)$  redefinition of  $\phi^j$  affects  $\bar{\beta}^{ij}$  not only for a given  $i$  but for every value of the index  $i$ . In these conditions we say that the two families of fields are related by a ‘weak gauge transformation’. These observations naturally lead us to introduce the notion of *weak gauge equivalence* for the

possible configurations  $\{A^i, \phi^i, \psi^i\}_{i \in I}$ , of the system formed by the gauge field and the charged matter fields.

Let us point out that the weak gauge principle could also be hinted from a study of the symmetries of the Lagrangian of the Standard Model of particle physics. In this theory, matter fields interact through the mediation of vector bosons, as a consequence the Lagrangian is invariant under weak gauge transformations, that is, there is a global  $U(1)$  symmetry associated with each matter field. On the contrary in a generic Lagrangian, with mixed neutral interaction terms as, say,  $\psi^3(\phi^*)^2$ , with  $3q_\psi - 2q_\phi = 0$ , there is only a global  $U(1)$  symmetry that involves all the matter fields at the same time, that is  $\alpha = \bar{\alpha}$ . The weak gauge principle naturally embodies the  $U(1)$  matter field-dependent redefinition freedom of the Standard Model, whereas the strong gauge principle does not.

In this context the weak gauge principle establishes the ‘intersection rule’ that expresses the relationship between fields defined on any two overlapping open sets of the covering. From a mathematical point of view the intersection rule says that the charged matter fields are sections of Hermitian complex line bundles (or equivalently principal  $U(1)$ -bundles). However, let us point out that the gauge field  $\{A^i\}_{i \in I}$  does not determine, in general, a principal  $U(1)$ -bundle. This is due to the following two reasons:

- (a) On the first place, the bundle may not exist at all. The family of 2-forms  $\{dA^i\}_{i \in I}$  determines a global closed 2-form  $F$  on  $M$ . It is well known that one can only associate with it a  $U(1)$ -principal bundle under appropriate integrality conditions.
- (b) Second, even if the principal  $U(1)$ -bundle exists, it might not be unique, since it is only determined up to flat bundles on  $M$ . It is well known that the family of isomorphism classes of flat  $U(1)$ -bundles is parametrized by the cohomology group  $H^1(M, U(1))$ . In particular the non-trivial flat bundles are in one-to-one correspondence with the torsion elements of  $H^2(M, \mathbb{Z})$ .

But even for the cases in which a principal bundle does exist, the weak gauge principle requires to consider the whole family of principal  $U(1)$ -bundles determined by the gauge field. In fact, given a pair of charged fields  $\psi, \phi$ , each of them determines a line bundle  $P_\psi, P_\phi$ . Now the weak gauge principle is equivalent, in this case, to saying that the line bundle  $P_\psi$  differs from  $P_\phi$  by a flat line bundle  $K$ . That is  $P_\psi = K \otimes P_\phi$ , where  $K$  represents, in geometrical terms, the holonomy difference of these two line bundles.

From a physical point of view, the class  $[K] \in H^1(M, U(1))$  measures the relative Aharonov–Bohm interference of  $\psi$  and  $\phi$ , which controls the different behaviour of the particles under topological Aharonov–Bohm experiments. If this class is trivial the phenomenology reduces to the standard one given by the strong gauge principle.

As a consequence, the line bundles  $P_\psi, P_\phi$  which describe the charged fields are in general not associated with the same  $U(1)$ -bundle but to different members of the family of  $U(1)$ -bundles determined by the gauge field. If  $K$  is not trivial, this is in sharp contrast with the standard description which assumes that all the charged fields are associated with the same principal bundle. This is essentially the mathematical content of the weak gauge principle. Therefore, the weak gauge principle differs in an essential way from the ordinary gauge principle in spacetimes  $M$  with a non-vanishing torsion subgroup of  $H^2(M, \mathbb{Z})$ . To our best knowledge this fact has not been previously recognized in the literature.

In the paper we analyse in detail the implications that the weak gauge principle has for charge quantization. In order to make the paper accessible to a wider audience, we have chosen to carry out this analysis by means of Čech cocycles rather than using the geometrical theory of bundles. We start just from the beginning with a good covering so that the relevance of triple intersections and Čech cohomology becomes clear. In some sense we continue the

generalization of Wu and Yang's paper made by Horváthy [23] who studied the monopole in a generic covering by considering the charge quantization argument in a generic field configuration. Since we show explicitly how the involved bundles are defined, the physically oriented reader may find this approach particularly clarifying while the mathematically oriented reader may find interesting how the cocycle condition arises from physical considerations based on the gauge principle and the existence of a matter field.

We shall see that the simple existence of a non-exact electromagnetic field on spacetime implies the quantization of charges and therefore the magnetic monopole is just one of the many possible non-exact field configurations leading to charge quantization. This fact should not come as a surprise since it has been recognized a number of times since Wu and Yang's work (see also [3, 18, 21]) that the quantization of the electric charge is related to the classification of principal  $U(1)$ -bundles in terms of Chern classes.

However, a main difference of this paper with respect to previous works is the fact that on spacetimes with torsion in the second integral cohomology group, the non-triviality of the bundles associated with the charged fields does not necessarily imply charge quantization.

Another consequence of the theory for spacetimes with torsion is the fact that it gives rise to two natural quantization units that we identify with the electric quantization unit (realized inside the quarks) and with minus the electron charge. In this framework the colour charge can have a topological origin, with the number of colours being related to the order of the torsion subgroup of  $H^2(M, \mathbb{Z})$ .

We also point out that the quantization of charge may be due to a weak non-exact component of the electromagnetic field extended over cosmological scales if at those scales a non-trivial topology of the spacetime manifold arises. This component could have formed in the initial instants of the Universe when its topology acquired a final form. Then the expansion of the Universe would have decreased its magnitude making it undetectable in today's experiments.

Let us mention that recent discussions [9, 19] on the role of coverings in the deduction of Dirac's quantization condition led us to believe that a treatment of charge quantization in general coverings and for general spacetimes like ours could help to clarify the assumptions that stay at the heart of topological quantization.

In order to finish this introduction, we recall that over the years other approaches have been introduced in order to give an explanation of the quantization of the electric charge. They involve topological arguments [10, 25, 34, 35], geometric quantization [38, 30], path integral considerations [1], anomaly cancellations [2, 7, 13, 15–17], Kaluza–Klein theory [26], a particular analysis of the Aharonov–Bohm potentials [4], loop quantization [11] or a particular quantum theory of the electric charge [39]. The Dirac quantization condition can be derived from the quantization of the total angular momentum [20] and can be related to the associativity of finite translations [24, 31]. However, there have been also claims of inconsistency of Dirac's quantization condition in second quantization [22]. Moreover, Schwinger suggested that the magnetic charge should be an *even* integer of the Dirac unit [36, 37]. These arguments are, however, not generally accepted [20, 32, 33].

## 2. The gauge principle in Minkowski spacetime

The theory of connections was developed by mathematicians in the 50s and only in the 70s the relation with physics and with the gauge principle was fully realized. Nowadays the electromagnetic field is described by a connection on a principal  $U(1)$ -bundle and a charged particle field is regarded as a global section of an associated bundle which is constructed from a representation of  $U(1)$  on the vector space  $\mathbb{C}$ . Only particles having a quantized charge can

be implemented in this mathematical setting and, for this reason, to recast the Dirac original argument in modern terms means to justify why this mathematical description is essentially the only one available. Therefore, for the moment we forget about the mathematics of principal fibre bundles and connections and consider the simple case of Minkowski spacetime. We define the gauge principle and only later we move on to see what happens in spacetimes with non-trivial topology.

Let  $M$  be a spacetime with a metric  $(+ - - -)$  and the topology of  $\mathbb{R}^4$  (Minkowski spacetime). The electromagnetic field  $F$  is a closed 2-form field on  $M$ . The matter field of a particle is a mapping  $\psi : M \rightarrow \mathbb{C}$ . We shall denote matter fields with the letters  $\psi$  and  $\phi$ . We shall explicitly use two matter fields since the quantization of charge is a relation between the charges of two fields. The meaning of what follows would be therefore clearer working with two matter fields at the same time. Moreover, we shall need at least two matter fields in order to distinguish between the weak and the strong gauge principles (see below). It is understood that our study can be straightforwardly generalized to any number of fields.

Since, the spacetime is contractible there is a potential (1-form) field  $A$  such that  $F = dA$ ; moreover let  $A'$  be another potential, we have  $d(A - A') = 0$  and since  $M$  is contractible ( $M$  simply connected would suffice here) we have  $A' = A + d\alpha$ , where  $\alpha : M \rightarrow \mathbb{R}$  is a function.

By definition the triplets  $(A, \psi, \phi), (A', \psi', \phi')$  are related by a *weak gauge transformation* if there exist functions  $\alpha, \bar{\alpha}$  differing by a constant  $h = \alpha - \bar{\alpha}$  and such that

$$A' = A + d\alpha, \quad (1)$$

$$\psi' = e^{iq_\psi \alpha} \psi, \quad (2)$$

$$\phi' = e^{iq_\phi \bar{\alpha}} \phi. \quad (3)$$

If  $h = 0$  the two triplets are related by a *strong gauge transformation*. The constant  $q_\psi$  (respectively  $q_\phi$ ) is the electric charge of the matter field  $\psi$  (respectively  $\phi$ ). The charges are quantized if there exists  $q \in \mathbb{R}^+$  such that  $q_\psi = mq, q_\phi = nq$  with  $n, m \in \mathbb{Z}$  coprime. Note that if  $q$  exists then it is defined univocally by the requirement that  $m$  and  $n$  are coprime (i.e. there are integers  $M, N$  such that  $Mm + Nn = 1$ ). Note also that by *charges* we shall always mean those parameters that appear in the gauge transformation. In general the Lagrangian depends on them and therefore they will have some experimental consequences from which their values can be recovered. We stress that we do not define the *charge* as the integral over a suitable surface of the 0-component of a certain conserved current. We also stress that the word *quantization* is used as a synonym for *discretization*. Making this choice we have followed the most used, although sometimes misleading, terminology.

Note that in most treatments only one field is considered and therefore the difference between weak and strong gauge transformations cannot be appreciated. Note also that the definition of weak gauge transformation is symmetric between  $\psi$  and  $\phi$  since  $\alpha$  and  $\bar{\alpha}$  differ by a constant and therefore  $\alpha$  can be replaced by  $\bar{\alpha}$  in equation (1). In the presence of many fields one would need a different function  $\alpha$  for each field. In the following the calculations will make sense in both the weak and strong cases.

There is no universally accepted definition of gauge principle but the following definition seems to summarize the most relevant features. A physical theory of the electromagnetic field satisfies the gauge principle if

**(Weak/strong) Gauge principle.** *The physical states of the theory are the equivalence classes  $[A, \psi, \phi]$  where two elements  $(A', \psi', \phi'), (A, \psi, \phi)$  are equivalent,  $(A', \psi', \phi') \sim (A, \psi, \phi)$ , iff they are related by a (weak, strong) gauge transformation for a suitable function(s)  $\alpha$ . The set of physical states will be denoted  $\{[A, \psi, \phi]\}$ .*

The difference between weak and strong gauge principles is not important in the Minkowskian case since the Lagrangian is usually invariant under symmetry transformations that multiply a field by a constant phase while keeping all the other fields constant. However, the difference will be relevant in a non-contractible topology. This feature has been overlooked in the past. Indeed, in the physical literature the gauge principle only appears in its strong version.

The gauge principle has been generalized to non-Abelian groups and has found extensive applications in quantum field theory. In the Standard Model the electromagnetic gauge transformation is imbedded in a non-trivial way in the electroweak gauge group  $SU(2) \otimes U(1)$ . In this case we can think of the  $U(1)$  gauge invariance of the present work as the  $U(1)$  sector of the electroweak group. Indeed, the quantization of the electric charge follows from the quantization of the  $U(1)$  sector (its charge is usually denoted with the letter  $Y$ ). For simplicity, but without loss of generality, we shall ignore this aspect here. We shall refer to the  $U(1)$  gauge invariance as the electromagnetic gauge invariance and to the constants  $q_\psi, q_\phi$ , as the electric charges.

We also stress that in the whole work we shall remain in a classical field theory approach and we will never be involved with the quantum theory. More precisely, the only quantum feature that we shall use is that of considering a particle as mathematically represented by a wavefunction rather than by a worldline; the dynamics, however, will remain completely classical, i.e. dictated by a Lagrangian and we will never use second quantization procedures. Moreover we shall never be involved with explicit expressions for the action or the observables. Our assumption is that the action and the observables have values dependent only on the physical state and not on its representant. Therefore they should be *gauge invariant*.

### 3. The gauge principle in a non-trivial topology

Let  $M$  be a curved spacetime with a metric  $g_{\mu\nu}$  of signature  $(+---)$ . Let  $F$  be a closed 2-form field on  $M$  which we do not assume to be contractible. Let  $\{U_i\}$  be a good covering of  $M$  i.e. the open sets  $U_\alpha$  are contractible and so are their arbitrary intersections [8, p 28]. We denote by  $U_{ij} = U_i \cap U_j$ ,  $U_{ijk} = U_i \cap U_j \cap U_k$ , the double and triple intersections, respectively. Note that the Latin index refers to the open set considered while the spacetime tensor indices are denoted with Greek letters. We have no compelling experimental evidence that our spacetime is not contractible thus we have in fact no evidence that more than one open set are required to cover  $M$ . Nevertheless, new investigations on the topology of the Universe had appeared in recent years [27, 28] and even some evidence of a non-trivial cosmological topology has been pointed out [29]. The topological argument explores the consequences of a non-contractible topology assumption. We shall give here a form to the original argument given by Dirac that is clearly related to the concept of Čech cohomology. In fact the Čech cohomology studies if a principal  $U(1)$  fibre bundle can be constructed over  $M$  and the Dirac argument essentially implies that the conditions that allow its existence are fulfilled.

In the previous section we have introduced the gauge principle. It is natural to generalize it to the present situation where we have many open sets  $U_i$ . First, we can repeat the same steps as before and see that in each open set  $U_i$ , given  $F$  on  $M$ , we have a potential  $A^i$ . Thus we can define what is a gauge transformation in each  $U_i$ . The same holds for a generic contractible open set  $U \subset M$  for which a potential  $A^U$  can be defined. In general if the set considered belongs to the good covering we denote the superscript with  $i$  in place of  $U$ . Here the contractible open sets play the role of the contractible spacetime considered previously. Note that there are special fields that are directly observable like  $F$  and  $g$  while others are not, like the electromagnetic fields  $A^i$  that receive for this reason an index of the open set where

they are defined. The reason is that in principle we may allow some discontinuity between  $A^i$  and  $A^j$  in  $U_{ij}$  as only the continuity of observable quantities really matters.

It remains the question of the observability of matter fields. If they are directly observable then we should not add to them an index corresponding to the open set. However, we want to reproduce the previously stated gauge principle in a given open set  $U$  and from that we already know that  $\psi$  cannot be observed completely because of the gauge principle. Thus we add an index  $i$ , and write  $\psi^i$  in correspondence of the open set  $U_i$ . In general the field will have a representant  $\psi^U$  in each contractible open set  $U$  and we shall regard the triplet  $(A, \psi, \phi)$  as the collection  $\{(A^U, \psi^U, \phi^U)\}$  for  $U \subset M$  contractible. Given the matter fields and the potential on each open set  $U$ , we shall use the notation  $(A, \psi, \phi) \equiv \{(A^U, \psi^U, \phi^U)\}$ .

We define a weak gauge transformation on  $U_i$  as

$$A'^i = A^i + d\alpha^i, \quad (4)$$

$$\psi'^i = e^{iq_\psi \alpha^i} \psi^i, \quad (5)$$

$$\phi'^i = e^{iq_\phi \bar{\alpha}^i} \phi^i, \quad (6)$$

where  $h^i = \alpha^i - \bar{\alpha}^i$  is a constant, and analogously for more general contractible open sets  $U$  not belonging to  $\{U_i\}$ . The strong gauge transformation satisfies  $h^i = 0$ . We shall also say that the fields  $(A', \psi', \phi')$  and  $(A, \psi, \phi)$  are gauge related in  $U$ . Then the (weak/strong) gauge principle is generalized as

**(Weak/strong) Gauge principle.** *The configurations are those collections  $\{(A^U, \psi^U, \phi^U)\}$  such that if  $U \subset V$ ,  $(A^V, \psi^V, \phi^V)|_U \sim (A^U, \psi^U, \phi^U)$  where  $\sim$  means that the fields between brackets are gauge related.*

*The physical states of the theory are the equivalence classes  $[A, \psi, \phi]$  of configurations, where two configurations  $(A', \psi', \phi')$ ,  $(A, \psi, \phi)$  are equivalent,  $(A', \psi', \phi') \sim (A, \psi, \phi)$ , iff for each open set  $U$  they are related by a gauge transformation for a suitable function  $\alpha^U$ .*

The set of physical states will be denoted by  $S = \{[A, \psi, \phi]\}$ .

The specification of a configuration  $(A^i, \psi^i, \phi^i)$  in the good covering  $\{U_i\}$  determines uniquely the state since it determines a representant in each contractible open subset.

**Remark 3.1.** If the charges are quantized  $q_\psi = mq$ ,  $q_\phi = nq$  with  $m$  and  $n$  coprime then, since what really matters is the phase factor in the gauge transformation, the physics is determined by the class whose elements are related by  $\alpha'^i = \alpha^i + a^i$  with  $a^i \in \frac{2\pi}{mq}\mathbb{Z}$  (and analogously for  $\bar{\alpha}^i$  and  $\bar{a}^i$ ). In the strong case since  $\alpha^i = \bar{\alpha}^i$ ,  $a^i = \bar{a}^i$ , in the end we have  $a^i \in \frac{2\pi}{q}\mathbb{Z}$ . In the weak case  $\alpha$  and  $\bar{\alpha}$  can be redefined freely as above and therefore  $h^i \in \mathbb{R}/(\frac{2\pi}{mnq}\mathbb{Z})$ .

If the charges are not quantized and we are in the strong case  $\alpha^i \in \mathbb{R}$ ,  $a^i = \bar{a}^i = 0$ .

If we are in the weak case, regardless of the quantization of charges,  $\alpha^i \in \mathbb{R}/(\frac{2\pi}{q_\psi}\mathbb{Z})$ ,  $a^i \in \frac{2\pi}{q_\psi}\mathbb{Z}$ , and analogously for  $\bar{\alpha}^i$ ,  $\bar{a}^i$ . Denoting by  $\Lambda$  the additive group given by the finite linear integer combinations of the real numbers  $2\pi q_\psi^{-1}$ ,  $2\pi q_\phi^{-1}$ , we have  $h^i \in \mathbb{R}/\Lambda$ . The quantized case considered previously is therefore a particular case of this one.

### 3.1. Intersection rule

The gauge principle implies that for a given configuration if  $U, V \subset M$ ,  $(A^V, \psi^V, \phi^V)|_{U \cap V} \sim (A^U, \psi^U, \phi^U)|_{U \cap V}$ . Let  $U = U_i$  and  $V = U_j$  then  $U \cap V = U_{ij}$  and we find that there are functions  $\beta^{ij}(\bar{\beta}^{ij}) : U_{ij} \rightarrow \mathbb{R}$  such that

$$A^i = A^j + d\beta^{ij}, \quad (7)$$



$$\psi^i = e^{iq_\psi \beta^{ij}} \psi^j, \tag{8}$$

$$\phi^i = e^{iq_\phi \bar{\beta}^{ij}} \phi^j, \tag{9}$$

where  $k^{ij} = \beta^{ij} - \bar{\beta}^{ij}$  is a constant which vanishes in the strong case. The first equation follows necessarily from the fact that both potentials have the same exterior differential in  $U_{ij}$ , the remaining two require the gauge principle. We shall refer to the system (7)–(9) as the *intersection rule* since it takes place in the intersection  $U_{ij}$  and relates fields from different sets.

Without loss of generality we can assume  $\beta^{ij} = -\beta^{ji}$  ( $\bar{\beta}^{ij} = -\bar{\beta}^{ji}$ ). Indeed equation (7) for the pairs  $(i, j)$  and  $(j, i)$  implies that  $\beta^{ji}$  and  $-\beta^{ij}$  differ by a constant and thus we can use the latter instead of the former in equation (7) for the pair  $(j, i)$ . Analogously equation (8) for the pairs  $(i, j)$  and  $(j, i)$  implies that  $e^{iq_\psi \beta^{ji}} = e^{-iq_\psi \beta^{ij}}$ , thus again  $-\beta^{ij}$  can be used in place of  $\beta^{ji}$  and analogously for  $\phi$ .

**Remark 3.2.** If the charges are quantized  $q_\psi = mq, q_\phi = nq$  with  $m$  and  $n$  being coprime then, since what really matters is the phase factor in the intersection rule, the physics is determined by the class whose elements are related by  $\beta^{ij} = \beta^{ij} + o^{ij}$  with  $o^{ij} \in \frac{2\pi}{mq} \mathbb{Z}$  (and analogously for  $\bar{\beta}^{ij}$  and  $\bar{o}^{ij}$ ). In the strong case since  $\beta^{ij} = \bar{\beta}^{ij}, o^{ij} = \bar{o}^{ij}$  we have  $o^{ij} \in \frac{2\pi}{q} \mathbb{Z}$ . In the weak case  $\beta$  and  $\bar{\beta}$  can be redefined freely as above and therefore  $k^{ij} \in \mathbb{R} / (\frac{2\pi}{mnq} \mathbb{Z})$ .

If the charges are not quantized and we are in the strong case then  $\beta^{ij} \in \mathbb{R}$ , and  $o^{ij} = \bar{o}^{ij} = 0$ .

If we are in the weak case, regardless of the quantization of charges,  $\beta^{ij} \in \mathbb{R} / (\frac{2\pi}{q_\psi} \mathbb{Z}), o^{ij} \in \frac{2\pi}{q_\psi} \mathbb{Z}$ , and analogously for  $\bar{\beta}^{ij}, \bar{o}^{ij}$ . Moreover,  $k^{ij} \in \mathbb{R} / \Lambda$ . The quantized case considered previously is therefore a particular case of this one.

Assume that the physical theory considered satisfies the gauge principle and hence the intersection rule. The actual physical problem depends both on  $F$  and on the functions  $\beta^{ij}$  (and  $\bar{\beta}^{ij}$ ) that appear in the intersection rule. Indeed  $\beta^{ij}$  does not depend solely on  $F$  and on the gauge choice of  $A^i$  in each set. Let us comment this more extensively. First note that  $\beta^{ij}$  changes under a gauge transformation on each  $U_i, A^i = A^i + d\alpha^i$  as

$$\beta^{ij} = \beta^{ij} + \alpha^i - \alpha^j. \tag{10}$$

Now fix the gauge, i.e. a choice of  $A^i$  in each set. This choice fixes  $\beta^{ij}$  only up to an additive constant, that is, the knowledge of  $F$  and the assumption that the intersection rule (7) holds do not determine  $\beta^{ij}$  completely. Thus the physical problem is determined both by  $F$  and, given a choice of gauge in each set, by a  $\beta^{ij}$  compatible with that choice (i.e., satisfying (7)). We shall call that  $\beta^{ij}$  *the physical  $\beta^{ij}$*  since suitable physical experiments may measure its value up to transformations  $\beta^{ij} = \beta^{ij} + o^{ij}$  with  $o^{ij}$  ranging in a suitable domain (remark 3.2) for a given gauge choice.

This is exactly what happens in the Aharonov–Bohm effect. Let  $\gamma$  be a closed curve on spacetime. Choose events  $e$  on the curve and on suitable intersections  $U_{ij}$  in such a way that the curve  $\gamma$  between two successive events along the curve lies entirely in the open set  $U_i$ . For convenience relabel the sets in such a way that  $i$  takes the successive values  $i = 1, \dots, N$  and identify  $N + 1$  with 1. Denote by  $e_i$  the successive events,  $e_i \in U_{i-1i}$ . The interference of a matter field with itself in an Aharonov–Bohm experiment along the closed curve  $\gamma$  is determined by the gauge invariant quantity which does not depend on the choice of  $\{e_i\}$  [40, 1]

$$\Phi[\gamma] = \exp \left\{ iq_\psi \sum_i \left[ \left( \int_{\gamma e_i}^{e_{i+1}} A^i \right) + \beta^{ii-1}(e_i) \right] \right\}. \tag{11}$$

Clearly the functions  $\beta^{ij}$  are not defined up to an arbitrary constant, otherwise the interference would be arbitrary. Indeed, they are defined, in a given gauge, only up to terms  $\frac{2\pi}{q_\psi}\mathbb{Z}$ . Therefore there is some more physical information encoded in the functions  $\beta^{ij}$  than what is determined alone by equation (7). If  $\gamma$  is a boundary,  $\gamma = \partial\Gamma$ , it is not difficult to show that

$$\sum_i \left[ \left( \int_{\gamma e_i}^{e_{i+1}} A^i \right) + \beta^{i i-1}(e_i) \right] = \int_\Gamma F - \sum_{U_{ijk}: \Gamma \cap U_{ijk} \neq \emptyset} c_{ijk} \tag{12}$$

where in the next section  $c_{ijk} = \beta^{ij} + \beta^{jk} + \beta^{ki}$  will be proved to be constant coefficients belonging to  $\frac{2\pi}{q_\psi}\mathbb{Z}$ . As a consequence if  $\gamma$  is contractible to a point

$$\Phi[\gamma] = \exp \left\{ i q_\psi \int_\Gamma F \right\}, \tag{13}$$

which is the better-known, but less general, expression for the interference phase. In a contractible topology the weak gauge principle would therefore lead to the usual phenomenology, i.e. the interference phase measured in the Aharonov–Bohm experiment would have the usual expression. However, in a truly non-contractible spacetime there does not exist an *a priori* constraint for the physical  $\beta^{ij}$  and therefore  $k^{ij}$  may differ from zero leading to a topological Aharonov–Bohm interference even for neutral particles (see section 5). This observation clarifies that strong and weak gauge principles can be distinguished in a spacetime having non-contractible topology. Since at present, the topology of the Universe has not yet been determined, we cannot yet rule out the weak gauge principle possibility.

Summarizing we can say that the physics is determined by a class  $[A^i, \beta^{ij}, \bar{\beta}^{ij}, \psi^i, \phi^i]$  (each term is regarded as a set of maps, for instance  $\beta^{ij}$  represents the set of maps  $\beta^{ij} : U_{ij} \rightarrow \mathbb{R}$ ) satisfying the intersection rule, and where two elements in the class are related by

$$(A^i, \beta^{ij}, \bar{\beta}^{ij}, \psi^i, \phi^i) \sim (A^i + d\alpha^i, \beta^{ij} + \alpha^i - \alpha^j, \bar{\beta}^{ij} + \bar{\alpha}^i - \bar{\alpha}^j), e^{iq_\psi\alpha^i} \psi^i, e^{iq_\psi\bar{\alpha}^i} \phi^i$$

with  $\alpha^i$  suitable functions (note that  $k^{ij} = k^{ij} + h^i - h^j$ ).

#### 4. The topological argument

Let us come to the core of the topological argument. We consider here only the field  $\psi$  but analogous considerations hold for  $\phi$ . The reader will recognize the relation with the Čech cohomology.

Consider a triple intersection  $U_{ijk}$ . We already know that in that set there is a function  $\beta^{ij}$  such that  $A^i = A^j + d\beta^{ij}$  and analogously for the pairs  $(i, k), (j, k)$ . Moreover, summing up the three equations just obtained we get

$$d(\beta^{ij} + \beta^{jk} + \beta^{ki}) = 0 \implies c_{ijk} = \beta^{ij} + \beta^{jk} + \beta^{ki} = \text{const} \text{ on } U_{ijk}. \tag{14}$$

The reader familiar with the Čech cohomology may realize that the constants  $c_{ijk}$  define a class in the Čech cohomology group  $H^2(M, \mathbb{R})$ , two elements in the same class being related by  $c'_{ijk} = c_{ijk} + o^{ij} + o^{jk} - o^{ik}$  for suitable constants  $o^{ij}$ . In fact, there is an isomorphism between the de Rham cohomology group  $H^2_{\text{dR}}(M, \mathbb{R})$ , to which  $[F]$  belongs, and the Čech cohomology group  $H^2(M, \mathbb{R})$  that sends  $[F]$  to  $[c_{ijk}]$ . However, note that we are identifying a precise representant of  $[c_{ijk}]$ , the *physical*  $c_{ijk}$ , thanks to the information that comes from the physical  $\beta^{ij}$  i.e., the one that satisfies all the intersection equations and not only (7). Repeating the same calculations for  $\phi$  we find that

$$c_{ijk} - \bar{c}_{ijk} = k^{ij} + k^{jk} + k^{ki}, \tag{15}$$

and therefore  $[c_{ijk}] = [\bar{c}_{ijk}]$  as classes belonging to  $H^2(M, \mathbb{R})$  even in the weak gauge principle case although the actual representant may change from matter field to matter field.

**Remark 4.1.** Consider the special case in which the charges are quantized. The physical  $\beta^{ij}$  is itself undetermined. Thus changing  $\beta^{ij} = \beta^{ij} + o^{ij}$  we obtain  $c'_{ijk} = c_{ijk} + o^{ij} + o^{jk} + o^{ki}$ . In the strong case  $o^{ij} \in (2\pi/q)\mathbb{Z}$  thus the intersection rules (7), (8) and (9) determine  $[qc_{ijk}] \in Z^2(M, \mathbb{R})/B^2(M, 2\pi\mathbb{Z})$  while  $[F]$  alone only determines  $[c_{ijk}] \in H^2(M, \mathbb{R})$ . In the weak case  $o^{ij} \in (2\pi/mq)\mathbb{Z}, \delta^{ij} \in (2\pi/nq)\mathbb{Z}$  thus  $[mqc_{ijk}] \in Z^2(M, \mathbb{R})/B^2(M, 2\pi\mathbb{Z})$  and analogously for  $[nq\bar{c}_{ijk}]$ . Moreover, as elements of  $Z^2(M, \mathbb{R})/B^2(M, 2\pi\mathbb{Z})$  the classes  $n[mqc_{ijk}]$  and  $m[nq\bar{c}_{ijk}]$  are in general different.

If the charges are not quantized and we are in the strong case  $[c_{ijk}], [\bar{c}_{ijk}] \in Z^2(M, \mathbb{R})$ .

If the charges are not quantized and we are in the weak case  $[q_\psi c_{ijk}] \in Z^2(M, \mathbb{R})/B^2(M, 2\pi\mathbb{Z})$  and analogously for  $[q_\phi \bar{c}_{ijk}]$ .

The topological argument goes on to prove that  $c_{ijk}$  are integer constants up to a common factor. Indeed, let us use repeatedly the intersection rule for  $\psi$  on  $U_{ijk}$

$$\begin{aligned} \psi^i &= e^{iq_\psi \beta^{ij}} \psi^j = e^{iq_\psi (\beta^{ij} + \beta^{jk})} \psi^k \\ &= e^{iq_\psi (\beta^{ij} + \beta^{jk} + \beta^{ki})} \psi^i = e^{iq_\psi c_{ijk}} \psi^i. \end{aligned} \tag{16}$$

Since this equation holds for any field  $\psi$  we have

$$q_\psi c_{ijk} = 2\pi m_{ijk}, \tag{17}$$

with  $m_{ijk} \in \mathbb{Z}$ . Repeating the argument for  $\phi$ ,

$$q_\phi \bar{c}_{ijk} = 2\pi n_{ijk}, \tag{18}$$

with  $n_{ijk} \in \mathbb{Z}$ , and using the relation between  $c_{ijk}$  and  $\bar{c}_{ijk}$  we obtain (we assume the charges to be different from zero, otherwise the issue of the quantization of charge would have a trivial affirmative answer)

$$\frac{2\pi}{q_\psi} m_{ijk} - \frac{2\pi}{q_\phi} n_{ijk} = k^{ij} + k^{jk} + k^{ki}. \tag{19}$$

We are going to separate the study into the strong and the weak cases. Although, the former will turn out to be a special case of the latter we shall consider them separately.

#### 4.1. The strong case

In the strong case ( $k^{ij} = 0$ ) the previous equation implies that if for a certain  $U_{ijk}, \bar{c}_{ijk} \neq 0$  (and hence  $n_{ijk} \neq 0$ ) then  $q_\psi/q_\phi$  is rational and therefore the charges are quantized. In this case there are  $m, n$  coprime integers such that  $q_\psi = mq, q_\phi = nq$

$$\frac{q_\psi}{q_\phi} = \frac{m_{ijk}}{n_{ijk}} = \frac{m}{n}, \tag{20}$$

thus  $m_{ijk}/m \in \mathbb{Z}, n_{ijk}/n \in \mathbb{Z}$  and hence  $qc_{ijk} \in 2\pi\mathbb{Z}$ . Moreover, since the charges are quantized the physical  $\beta^{ij}$  is determined only up to transformations  $\beta^{ij} = \beta^{ij} + o^{ij}$  with  $o^{ij} \in \frac{2\pi}{q}\mathbb{Z}$  and hence  $[qc_{ijk}] \in H^2(M, 2\pi\mathbb{Z})$  or  $[\frac{qF}{2\pi}] \in H^2(M, \mathbb{Z})$ . If for any  $U_{ijk}, c_{ijk} = 0$  one cannot infer whether the charges are quantized or not, and in the former case, by the same argument used above,  $[qc_{ijk}] \in H^2(M, 2\pi\mathbb{Z})$  is the trivial class.

Thus we have proved that there are only two possibilities:

- (A1) The charges are not quantized and for any  $U_{ijk}$  the physical  $c_{ijk}$  satisfies  $c_{ijk} = 0$ .
- (A2) The charges are quantized and  $[qc_{ijk}] \in H^2(M, 2\pi\mathbb{Z})$  (i.e.,  $[\frac{q}{2\pi}F] \in H^2(M, \mathbb{Z})$ ).

These two possibilities express the old Dirac's quantization argument in the language of Čech cohomology. Note that in both cases there exists a principal  $U(1)$ -bundle  $Q$  with transition functions  $g_{ij} = e^{iq\beta^{ij}}$ , since  $qc_{ijk} \in 2\pi\mathbb{Z}$ . The matter fields can then be regarded as sections of bundles associated with  $Q$  through representations  $\rho : U(1) \rightarrow GL(1, \mathbb{C})$ , where  $\rho$  is  $u \rightarrow u^m$  for  $\psi$  and  $u \rightarrow u^n$  for  $\phi$ .

If  $F$  is exact this mathematical setting for the matter fields is not compulsory. Indeed, in that case there are constants  $b^{ij} \in \mathbb{R}$  such that

$$(\beta^{ij} - b^{ij}) + (\beta^{jk} - b^{jk}) + (\beta^{ki} - b^{ki}) = 0. \quad (21)$$

Therefore, there exists a  $(\mathbb{R}, +)$ -principal bundle  $R$  with transition functions  $g_{ij} = \beta^{ij} - b^{ij}$ . The matter fields can then be regarded as sections of vector bundles associated with  $R$ .

#### 4.2. The weak case

In the weak case the deduction of the quantization condition is more involved. We said that  $[c_{ijk}] = [\bar{c}_{ijk}]$  as classes belonging to  $H^2(M, \mathbb{R})$ . This should be expected since the isomorphism between  $H^2(M, \mathbb{R})$  and  $H_{\text{dR}}^2(M, \mathbb{R})$  associates with both  $[c_{ijk}]$ ,  $[\bar{c}_{ijk}]$  the class  $[F]$  of the electromagnetic field. By remark 3.2 and equation (19) it follows that  $[k^{ij}] \in H^1(M, \mathbb{R}/\Lambda)$ . We mention that  $H^1(M, \mathbb{R}/\Lambda)$  is the set of isomorphism classes of principal  $\mathbb{R}/\Lambda$ -bundles. For a treatment of  $\mathbb{R}/\Lambda$ -bundles we refer the reader to [8, section 2.5].

There are two cases:

(1) If the class  $[c_{ijk}] = [\bar{c}_{ijk}] \in H^2(M, \mathbb{R})$  is not trivial ( $F$  is not exact) by Poincaré duality we can find a closed surface  $S$  such that  $\int_S F \neq 0$ . It can also be shown that this integral can be expressed as the sum of some coefficients  $c_{ijk}$  such that  $U_{ijk} \cap S \neq \emptyset$  (see for instance [1]) and analogously for  $\bar{c}_{ijk}$ . The equations (17) and (18) imply

$$q_\psi \int_S F \in 2\pi\mathbb{Z}, \quad (22)$$

$$q_\phi \int_S F \in 2\pi\mathbb{Z}, \quad (23)$$

and therefore the charges are quantized. Substituting  $q_\psi = mq$ ,  $q_\phi = nq$  and using the fact that  $m$  and  $n$  are coprime we obtain  $\int_S F \in \frac{2\pi}{q}\mathbb{Z}$ , that is  $[\frac{q}{2\pi}F] \in H^2(M, \mathbb{Z})$ . From equations (18) and (15)

$$nc_{ijk} = \frac{2\pi}{q}n_{ijk} + n(k^{ij} + k^{jk} + k^{ki}). \quad (24)$$

Using equation (17) we obtain ( $nN + mM = 1$ )

$$c_{ijk} = \frac{2\pi}{q}(Nn_{ijk} + Mm_{ijk}) + Nn(k^{ij} + k^{jk} + k^{ki}), \quad (25)$$

$$\bar{c}_{ijk} = \frac{2\pi}{q}(Nn_{ijk} + Mm_{ijk}) - Mm(k^{ij} + k^{jk} + k^{ki}). \quad (26)$$

Multiplying equation (15) by  $mn$  and using (17) and (18) we obtain

$$mnq(k^{ij} + k^{jk} + k^{ki}) \in 2\pi\mathbb{Z}, \quad (27)$$

therefore, as (remark 3.2)  $mnqk^{ij} \in \mathbb{R}/2\pi\mathbb{Z}$ , the coefficients  $mnqk^{ij}$  define a cocycle in the Čech cohomology. It has been previously pointed out that under gauge transformations  $k^{ij} = k^{ij} + h^i - h^j$ , thus

$$[mnqk^{ij}] \in H^1(M, \mathbb{R}/(2\pi\mathbb{Z})). \quad (28)$$

In other words, in the weak quantized case there exists a flat bundle  $K$  with transition functions  $e^{imnqk^{ij}}$ . We call the class  $k = [mnqk^{ij}]$ , the (relative) *interference class*.

From equation (27) it follows that  $mqc_{ijk} \in 2\pi\mathbb{Z}$  and analogously for  $\bar{c}_{ijk}$ . But we already know (remark 4.1) that  $[mqc_{ijk}] \in Z^2(M, \mathbb{R})/B^2(M, 2\pi\mathbb{Z})$  and thus

$$[mqc_{ijk}] \in H^2(M, 2\pi\mathbb{Z}), \tag{29}$$

$$[nq\bar{c}_{ijk}] \in H^2(M, 2\pi\mathbb{Z}). \tag{30}$$

These classes can be trivial or not, in any case we can construct two principal  $U(1)$ -bundles  $P_\psi$  and  $P_\phi$  with transition functions  $e^{imq\beta^{ij}}$  and  $e^{inq\bar{\beta}^{ij}}$ , respectively. Consider the short exact sequence

$$0 \rightarrow 2\pi\mathbb{Z} \rightarrow \mathbb{R} \xrightarrow{e^{ix}} U(1) \rightarrow 1, \tag{31}$$

where we identify  $U(1)$  and  $\mathbb{R}/2\pi\mathbb{Z}$ . It gives rise to the long exact sequence

$$\begin{aligned} 0 \rightarrow H^1(M, 2\pi\mathbb{Z}) \rightarrow H^1(M, \mathbb{R}) \xrightarrow{\sigma} \\ H^1(M, U(1)) \xrightarrow{\eta} H^2(M, 2\pi\mathbb{Z}) \xrightarrow{\gamma} H^2(M, \mathbb{R}) \rightarrow \dots, \end{aligned}$$

where  $\text{Im}(\eta) = \text{Ker}(\gamma)$  is the torsion subgroup of  $H^2(M, 2\pi\mathbb{Z})$ . Equation (15) multiplied by  $mn$  reads

$$n[mqc_{ijk}] - m[nq\bar{c}_{ijk}] = \eta([mnqk^{ij}]), \tag{32}$$

or

$$K = P_\psi^n \otimes P_\phi^{-m}. \tag{33}$$

If the manifold has vanishing torsion then  $K$  is trivial. One should be careful here because in general although

$$\eta([mnqk^{ij}])_{ijk} = mnq(k^{ij} + k^{jk} + k^{ki}), \tag{34}$$

this class is not necessarily trivial as  $mnqk^{ij}$  does not take values in  $2\pi\mathbb{Z}$ .

Consider the functions

$$q_{MN}^{ij} = Mm\beta^{ij} + Nn\bar{\beta}^{ij}, \tag{35}$$

where the index  $MN$  recalls that the constants  $M$  and  $N$  such that  $Mm + Nn = 1$  are not unique. If  $M'$  and  $N'$  is another pair with the same property then it is easy to show that there exists an integer  $j$  such that  $M' = M + jn$ ,  $N' = N - jm$ . Thus

$$q_{M'N'}^{ij} = q_{MN}^{ij} + jmnk^{ij}. \tag{36}$$

The redefinition freedom of  $\beta^{ij}$  and  $\bar{\beta}^{ij}$  (remark 3.2) implies that  $q_{MN}^{ij}(x) \in \mathbb{R}/\frac{2\pi}{q}\mathbb{Z}$ . Moreover, according to equations (25) and (26)

$$\xi_{MN}^{ijk} = q_{MN}^{ij} + q_{MN}^{jk} + q_{MN}^{ki} = \frac{2\pi}{q}(Nn_{ijk} + Mm_{ijk}) \in \frac{2\pi}{q}\mathbb{Z}$$

define a class  $[q\xi_{MN}^{ijk}] \in H^2(M, 2\pi\mathbb{Z})$  and the transition functions  $e^{iqq_{MN}^{ij}}$  define a principal  $U(1)$ -bundle  $Q_{MN}$ . equation (36) implies that

$$Q_{M'N'} = Q_{MN} \otimes K^j. \tag{37}$$

Since  $K$  belongs to the torsion,  $\gamma([q\xi_{MN}^{ijk}])$  does not depend on the choice  $M, N$ . From equation (29) we have

$$\gamma([mqc_{ijk}]) = mq[F] \in H^2(M, \mathbb{R}), \tag{38}$$

and from equations (28) and (25)

$$\gamma([mqc_{ijk}]) = m\gamma([q\xi_{MN}^{ijk}]), \tag{39}$$

so that

$$\gamma([q\xi_{MN}^{ijk}]) = q[F] \in H^2(M, \mathbb{R}). \tag{40}$$

In other words, the classes on  $H^2(M, 2\pi\mathbb{Z})$  associated with the principal bundles  $Q_{MN}$  all project on the electromagnetic field class on  $H^2(M, \mathbb{R})$ . We find again that  $[\frac{qF}{2\pi}]$  is an integer class.

The torsion subgroup is a finitely generated Abelian group. The interference class determines a flat bundle  $K$  such that  $K^{\text{ord}\eta(k)}$  is trivial, where  $\text{ord}\eta(k)$  is the order of the Abelian subgroup generated by the image  $\eta(k)$  of the interference class on  $H^2(M, 2\pi\mathbb{Z})$ . As a consequence

$$Q_{M+n\text{ord}\eta(k)N-m\text{ord}\eta(k)} = Q_{MN}, \tag{41}$$

and the principal bundle denoted by  $Q^{\text{ord}\eta(k)}$  and defined by

$$Q^{\text{ord}\eta(k)} \equiv Q_{MN}^{\text{ord}\eta(k)} \tag{42}$$

does not depend on the choice of  $M, N$ .

Now, note that  $m\beta^{ij} = mq_{MN}^{ij} + Nmkn^{ij}$  and  $n\bar{\beta}^{ij} = nq_{MN}^{ij} - Mmnk^{ij}$  so that

$$P_\psi = Q_{MN}^m \otimes K^N \tag{43}$$

$$P_\phi = Q_{MN}^n \otimes K^{-M} \tag{44}$$

and

$$Q_{MN} = P_\psi^M \otimes P_\phi^N \tag{45}$$

which hold for every pair  $M, N$ , satisfying  $Mm + Nn = 1$ . If  $\eta(k)$  is trivial then  $K$  is trivial,  $\text{ord}\eta(k) = 1$  and  $Q = Q_{MN}$  does not depend on the choice  $M, N$ . Moreover, equations (43), (44) show that the bundles  $P_\psi$  and  $P_\phi$  admit a well-defined root  $Q$  as in the strong case. Conversely, if a bundle  $Q$  exists such that  $P_\psi = Q^m$  and  $P_\phi = Q^n$  then equation (33) shows that  $K$  is trivial.

Assume that  $K$  is trivial. We already know that there is a root principal  $U(1)$ -bundle  $Q$ . Moreover, there are constants  $K^{ij} \in \frac{2\pi}{mnq}\mathbb{Z}$  such that

$$k^{ij} + k^{jk} + k^{ki} = K^{ij} + K^{jk} + K^{ki}.$$

Define  $o^{ij} = -NnK^{ij} \in \frac{2\pi}{mq}\mathbb{Z}$  and  $\bar{o}^{ij} = MmK^{ij} \in \frac{2\pi}{nq}\mathbb{Z}$ , so that  $K^{ij} = \bar{o}^{ij} - o^{ij}$ . Redefine  $\beta'^{ij} = \beta^{ij} + o^{ij}$  and analogously for  $\bar{\beta}'^{ij}$ , then due to remark 4.1, equation (15) can be written as

$$c'_{ijk} = \bar{c}'_{ijk}, \tag{46}$$

or

$$(\beta'^{ij} - \bar{\beta}'^{ij}) + (\beta'^{jk} - \bar{\beta}'^{jk}) + (\beta'^{ki} - \bar{\beta}'^{ki}) = 0$$

which can be regarded as the condition for the existence of a  $(\mathbb{R}, +)$  principal bundle. Since the fibre is contractible the bundle is trivial and hence there are functions  $\alpha^i(x)$  such that  $\beta'^{ij} = \bar{\beta}'^{ij} + \alpha^i - \alpha^j$  or  $k'^{ij} = \alpha^i - \alpha^j$ . A weak gauge transformation that sends  $k'^{ij}$  to zero exists iff  $\alpha^i = h^i + \alpha$  for a suitable function  $\alpha$  and for suitable constants  $h^i \in \mathbb{R}/(\frac{2\pi}{mnq}\mathbb{Z})$ . This condition is satisfied iff  $[mnqk'^{ij}] \in H^1(M, U(1))$  is trivial. Thus if

$[mnqk^{ij}] \in H^1(M, U(1))$  is trivial we are actually in the strong case as the condition  $k^{ij} = 0$  is preserved under strong gauge transformations.

(2) It could be that although  $[c_{ijk}] \in H^2(M, \mathbb{R})$  is trivial (i.e.  $[c_{ijk}] \in B^2(M, \mathbb{R})$ ) the charges are not quantized. In this case there are constants  $b^{ij} \in \mathbb{R}$  such that  $c_{ijk} = b^{ij} + b^{jk} + b^{ki}$  thus  $(\beta^{ij} - b^{ij}) + (\beta^{jk} - b^{jk}) + (\beta^{ki} - b^{ki}) = 0$ . The functions  $\beta^{ij} - b^{ij}$  can be regarded as the transition functions of a principal bundle of structure group  $(\mathbb{R}, +)$  as the previous condition states that the cocycle condition for the transition functions is satisfied. Since the fibre is contractible the principal bundle is trivial and therefore there are functions  $\alpha^i$  such that  $\beta^{ij} - b^{ij} + \alpha^i - \alpha^j = 0$ . This last equation means that there is a particular gauge in each  $U_i$  such that the functions  $\beta^{ij}$  become constant,  $\beta^{ij} = b^{ij} \in \mathbb{R}/\frac{2\pi}{q_\psi}\mathbb{Z}$  (where we have used the indeterminacy of  $\beta^{ij}$ ) and therefore, since the coefficients  $k^{ij}$  are constant,  $\bar{\beta}^{ij} = b^{ij} - k^{ij} \in \mathbb{R}/\frac{2\pi}{q_\phi}\mathbb{Z}$  are constant too. The equations (17), (18), provide further constraints

$$q_\psi(b^{ij} + b^{jk} + b^{ki}) = 2\pi m_{ijk}, \tag{47}$$

$$q_\phi(\bar{b}^{ij} + \bar{b}^{jk} + \bar{b}^{ki}) = 2\pi n_{ijk}. \tag{48}$$

We can associate with the constants  $b^{ij}$  (respectively  $\bar{b}^{ij}$ ) a flat bundle of transition functions  $e^{iq_\psi b^{ij}}$  (respectively  $e^{iq_\phi \bar{b}^{ij}}$ ) which can be trivial or not. From equation (47) it follows that  $[q_\psi c_{ijk}] \in H^2(M, 2\pi\mathbb{Z})$  and analogously for  $[q_\phi \bar{c}_{ijk}]$ . The classes of  $H^2(M, 2\pi\mathbb{Z})$  that are trivial when considered as classes of  $H^2(M, \mathbb{R})$  belong to the *torsion* of  $H^2(M, 2\pi\mathbb{Z})$ .

Note that if  $[q_\psi c_{ijk}], [q_\phi \bar{c}_{ijk}] \in H^2(M, 2\pi\mathbb{Z})$  are trivial then there are functions  $\gamma^i, \bar{\gamma}^i$  such that  $b^{ij} - \gamma^i + \gamma^j \in \frac{2\pi}{q_\phi}\mathbb{Z}$  and  $\bar{b}^{ij} - \bar{\gamma}^i + \bar{\gamma}^j \in \frac{2\pi}{q_\psi}\mathbb{Z}$ . These equations mean that the functions  $\beta^{ij}, \bar{\beta}^{ij}$  can be redefined so that there exist functions  $\alpha^i, \bar{\alpha}^i$  such that  $\beta^{ij} + \alpha^i - \alpha^j = 0, \bar{\beta}^{ij} + \bar{\alpha}^i - \bar{\alpha}^j = 0$ . In particular it is possible to find a weak gauge transformation such that  $e^{iq_\psi \beta^{ij}} = 1$  or  $e^{iq_\phi \bar{\beta}^{ij}} = 1$  but these gauges do not necessarily coincide. They coincide iff there exist constants  $h^i \in \mathbb{R}/\Lambda$  such that  $k^{ij} = h^i - h^j$  since in this case the terms  $h^i$  can be removed with a weak gauge transformation. Since  $[k^{ij}] \in H^1(M, \mathbb{R}/\Lambda)$  we conclude that the problem reduces to the strong case if  $[k^{ij}]$  is trivial.

Summarizing, there are two possibilities:

- (B1) The charges are not quantized,  $[k^{ij}] \in H^1(M, \mathbb{R}/\Lambda), [c_{ijk}] = [\bar{c}_{ijk}] \in H^2(M, \mathbb{R})$  is trivial and  $[q_\psi c_{ijk}], [q_\phi \bar{c}_{ijk}] \in H^2(M, 2\pi\mathbb{Z})$ . If these three classes are trivial we are in the strong case (A1).
- (B2) The charges are quantized and  $[mqc_{ijk}], [nq\bar{c}_{ijk}] \in H^2(M, 2\pi\mathbb{Z})$  satisfy  $\gamma([mqc_{ijk}]) = m[qF], \gamma([nq\bar{c}_{ijk}]) = n[qF]$  and  $[\frac{qF}{2\pi}]$  is an integer class. There are principal  $U(1)$ -bundles  $P_\psi$  and  $P_\phi$  associated with the classes  $[mqc_{ijk}], [nq\bar{c}_{ijk}]$ , that satisfy  $P_\psi^n \otimes P_\phi^{-m} = K$  where  $K$  is a flat bundle associated with the class  $\eta([mnqk^{ij}])$  where  $[mnqk^{ij}] \in H^1(M, U(1))$  is the interference class.  $K$  is trivial iff a root principal  $U(1)$ -bundle  $Q$  exists such that  $P_\psi = Q^m, P_\phi = Q^n$ . For every  $M, N$  such that  $Mm + Nn = 1$  equations (43), (44) and (45) hold. In particular the principal bundle defined by (42) does not depend on the choice of  $M, N$ . If  $[mnqk^{ij}] \in H^1(M, U(1))$  is trivial we are in the strong case (A2).

From the above study we conclude that in both the strong and weak cases if the electromagnetic field is not exact the charges are quantized (cases (A2) and (B2)). In any case, independently of the exactness of the electromagnetic field the most interesting case is (B2) as the charges are observationally quantized and (A2) is a special case of it.

A relevant difference between the weak and strong cases is that in the weak case there could be non-quantized charges with  $[q_\psi c_{ijk}] \in H^2(M, 2\pi\mathbb{Z})$  non-trivial and  $[q_\psi c_{ijk}] \in H^2(M, \mathbb{R})$



trivial. Such  $[q_\psi c_{ijk}]$  are non-trivial torsion classes and generate non-trivial flat bundles. If the weak gauge principle holds it is no longer true that a particle description through non-trivial bundles implies the quantization of charges. Moreover, the existence of a root principal bundle  $Q$  cannot be inferred in the weak case. The description of matter fields as sections of vector bundles associated with the same universal bundle then radically changes. Each particle has its own principal bundle.

The long exact sequence for the quantized case implies

$$\ker \eta = H^1(X, \mathbb{R})/H^1(X, \mathbb{Z}). \quad (49)$$

Note that in a simply connected spacetime  $\ker \eta = 0$  since  $H^1(X, \mathbb{R}) \sim H_{\text{dR}}^1(X, \mathbb{R}) = 0$  as in a simply connected manifold all the closed 1-forms are exact. This can be also seen from the universal covering theorem which states that  $H^1(X, A) \simeq \text{Hom}(\pi_1(X), A)$ , however one should be careful since  $H^1(X, A) = 0$  does not mean that  $\pi_1(X) = \{e\}$ . Moreover, in a simply connected spacetime the torsion of  $H^2(M, 2\pi\mathbb{Z})$  vanishes since it is the image of  $H^1(M, U(1))$  under  $\eta$ . We conclude that the strong case is equivalent to the weak case in simply connected manifolds.

## 5. Interpretation

We give an interpretation of (B2) which is the most interesting case from the physical point of view.

The generic matter field, say  $\psi$ , may not be described as the section of a vector bundle associated with  $Q$  under the representation  $\rho_\psi : U(1) \rightarrow GL(1, \mathbb{C}), u \rightarrow u^m$  since the root bundle  $Q$  does not always exist. On the contrary, each field has its own principal bundle, for instance  $\psi$  is a section of a vector bundle associated with  $P_\psi$  under the trivial representation  $u \rightarrow u$ . In general we can regard every field as a section of a vector bundle associated with a  $U(1)$ -bundle under the trivial representation. In this way the different particles are in one-to-one correspondence with the  $U(1)$  principal bundles. The possibility of describing the fields as sections of vector bundles associated with *the same* principal bundle under non-trivial representations arises only if the different  $U(1)$ -bundles considered have a common root. Note that on  $P_\psi$  a connection can be defined that takes, in suitable local coordinates the form,  $\omega_\psi = i(d\alpha^i - m q A^i)$ , so that covariant derivatives of matter fields make sense. However, no universal principal bundle  $Q$  with a universal connection of the form  $\omega = i(d\alpha^i - q A^i)$  as in usual (strong) gauge theory exists. In any case, on the principal bundles  $Q_{MN}$  a connection of that form can be defined although the principal bundle associated with the generic particle will not be always of the form  $Q_{MN}^a$  for suitable  $a$  and  $M, N$ .

Consider a particle obtained as a bound state of  $z_1$  particles  $\psi$  and  $z_2$  particles  $\phi$ . The numbers  $z_1$  and  $z_2$  are integers and if say  $z_2$ , is negative then there are  $|z_2|$  antiparticles of  $\phi$  in the bound state. The actual forces responsible for the bound state may not be of electromagnetic origin and are not important for our analysis. The new bound state is described by the principal bundle  $P_\psi^{z_1} \otimes P_\phi^{z_2}$ .

Equations (45) and (33) show that all the  $U(1)$  principal bundles of the form

$$P = Q_{MN}^a \otimes K^b \quad a \in \mathbb{Z}, \quad b \in \mathbb{Z}/\text{ord } \eta(k) \quad (50)$$

$$= P_\psi^{Ma+nb} \otimes P_\phi^{Na-mb} \quad (51)$$

can be generated from the physical building blocks  $P_\psi$  and  $P_\phi$ , however, no common root exists. Under the one-to-one identification of  $U(1)$  principal bundles and fields, the quantity  $aq$  represents the field charge. In particular there are  $\text{ord } \eta(k)$  neutral particles ( $a = 0$ ),

$$K^0, K^1, \dots, K^{\text{ord } \eta(k)-1} \quad (52)$$



which can also be regarded as different topological vacuum states. They form a group  $\langle K \rangle$  isomorphic to  $\mathbb{Z}/\text{ord } \eta(k)$  under tensorial multiplication. Under the same operation they act on  $H^2(M, 2\pi\mathbb{Z})$  separating it into cyclic orbits of  $\text{ord } \eta(k)$  elements each. For instance a principal bundle  $P_\psi$  whose class is in  $H^2(M, 2\pi\mathbb{Z})$  belongs to the orbit

$$P_\psi, P_\psi \otimes K, \dots, P_\psi \otimes K^{\text{ord } \eta(k)-1}. \tag{53}$$

The neutral content of a generic charged particle is not univocally determined. Indeed,

$$Q_{MN}^a \otimes K^b = Q_{M'N'}^a \otimes K^{b-ja}. \tag{54}$$

The interpretation becomes clear looking at equations (45) and (33). The particle represented by the principal bundle  $P$  may be regarded as containing  $aM$  particles  $\psi$ ,  $aN$  particles  $\phi$  and  $b$  neutral particles  $K$ . However, the particles  $\psi$  and  $\phi$  can change in number according to equation (33) as they can annihilate to form neutral particles  $K$ . The neutral particle content cannot in general be determined. However, if  $a$  is a multiple of  $\text{ord } \eta(k)$ ,  $a = \tilde{a} \text{ord } \eta(k)$ , then the constant  $b$  does not depend on the choice  $M, N$

$$P = (Q^{\text{ord } \eta(k)})^{\tilde{a}} \otimes K^b \quad \tilde{a} \in \mathbb{Z}. \tag{55}$$

Thus the particles having a charge multiple of  $\text{ord } \eta(k)q$  have a special role as they have a well-defined neutral particle content.

Given two fields  $\psi$  and  $\phi$ , the class  $k = [mnqk^{ij}] \in H^1(M, U(1))$  that we termed the (relative) *interference class* determines the different behaviour of the fields under Aharonov–Bohm interference caused by the topology of the Universe. In other words the topology of the Universe (i.e. its ‘holes’), being non-trivial, may act in a way analogous to the solenoid in the Aharonov–Bohm experiment. However, contrary to what could be naively expected from this analogy the interference phases of  $\psi$  and  $\phi$  are not of the form  $u^m, u^n$  for a suitable  $u \in U(1)$ . The interference class determines a different way in which these particles couple with the topology of the Universe.

We can see this fact easily from the expression of the Aharonov–Bohm phase for the neutral particle  $K$ . Using equation (11)

$$\begin{aligned} \Phi_k[\gamma] &= (\Phi_\psi[\gamma])^n (\Phi_\phi[\gamma])^{-m} = \exp \left\{ imnq \sum_i k^{ii-1} \right\} \\ &= \langle k, [\gamma] \rangle, \end{aligned} \tag{56}$$

where  $\langle \cdot, \cdot \rangle$  is the dual pairing between  $H^1(M, U(1))$  and  $H_1(M, U(1))$ , and  $[\gamma]$  is the homology class whose representant is  $\gamma$ .

A particular case is obtained if  $\eta(k)$  is trivial in  $H^2(M, 2\pi\mathbb{Z})$ , i.e. if  $K$  is trivial. In this case  $k = \sigma(k_{\mathbb{R}})$  for a suitable class  $k_{\mathbb{R}} \in H^1(M, \mathbb{R})$ . Then

$$\Phi_k[\gamma] = \langle k, [\gamma] \rangle = e^{i\langle k_{\mathbb{R}}, [\gamma]_{\mathbb{R}} \rangle_{\mathbb{R}}}, \tag{57}$$

where  $\langle \cdot, \cdot \rangle_{\mathbb{R}}$  is the dual pairing between  $H^1(M, \mathbb{R})$  and  $H_1(M, \mathbb{R})$ , and  $[\gamma]_{\mathbb{R}}$  is the corresponding homology class whose representant is  $\gamma$ . In other words there is a closed 1-form denoted again with  $k_{\mathbb{R}}$  such that  $\Phi_k[\gamma] = \exp \{ i \oint_{\gamma} k_{\mathbb{R}} \}$ . Thus there is a topological Aharonov–Bohm effect that acts on neutral particles. It acquires its characteristic exponential form only if the torsion of the particle vanishes.

### 6. Quarks

In the usual strong case there is only one charge quantization unit  $q$ . Since the quarks are described in the Lagrangian by matter fields, the quantization unit should be necessarily

identified with one third the (minus) electric charge  $q = e/3$  or with a subunit  $q/z, z \in \mathbb{Z}^+$ . It is therefore incorrect in the strong case to identify the quantum  $q$  with minus the electric charge  $e$ . However, it is an experimental fact that all the observed particles have a charge that is a multiple of  $e = 3q$  (quark confinement). The usual strong case does not provide a mathematics sufficiently rich to describe such a situation.

In the weak case we have seen that given two fields there are two quantization units of relevance for the theory. The charge quantum  $q$  and the unit  $\text{ord } \eta(k)q$ . It is natural to identify  $e \equiv \text{ord } \eta(k)q$ , and hence to assume that  $\eta(k)$  is a torsion class  $K$  that generates a cyclic subgroup of order three,  $\text{ord } \eta(k) = 3$ . Next we assume for simplicity that  $H^2(M, 2\pi\mathbb{Z})$  has a torsion subgroup which coincides with the cyclic subgroup  $\langle K \rangle$ .

As we have seen the cyclic group acts on the principal bundle of the field and generates an entire orbit of  $\text{ord } \eta(k) = 3$  particles. We proceed with the following identification. The fields considered are quarks and the elements of the orbits correspond to the different colours of the same particle flavour.

For instance, let the two fields be the ‘up’ and ‘down’ quarks for a certain colour. They have charge  $2q$  and  $-q$  respectively. Then there is a principal  $U(1)$ -bundle  $Q$  such that the following bundles are identified with those of  $u(r), u(g)$  and  $u(b)$

$$u: Q^2, \quad Q^2 \otimes K, \quad Q^2 \otimes K^2. \quad (58)$$

The actual identification of this bundles with the corresponding colour is important only up to cyclic permutations. Analogously the bundles corresponding to  $d(r), d(g)$  and  $d(b)$  are

$$d: Q^{-1}, \quad Q^{-1} \otimes K, \quad Q^{-1} \otimes K^2. \quad (59)$$

The next flavour generations  $(c, s)$  and  $(t, b)$  live on the same principal bundles. For instance  $c(r)$  is a section on the same bundle of  $u(r)$ . Here, we are considering the simplest possible model. There is enough room for many other possibility for instance considering a torsion subgroup larger than those considered here. To the antiparticles correspond the principal bundles

$$\begin{aligned} \bar{u}: Q^{-2}, \quad Q^{-2} \otimes K^2, \quad Q^{-2} \otimes K, \\ \bar{d}: Q, \quad Q \otimes K^2, \quad Q \otimes K, \end{aligned}$$

and analogously for  $(\bar{c}, \bar{s})$  and  $(\bar{t}, \bar{b})$ .

The  $SU(3)$  colour transformations are not ordinary matrices. Indeed, in order to preserve the above correspondences under colour transformations we must generalize these matrices. In this model the coefficients  $B_{ij}$  of a colour matrix  $B$  are not  $\mathbb{C}$  numbers, instead they are sections of a complex vector bundle associated with  $K^{2(j-i)}$ , that is, using the identification between  $U(1)$ -bundles and fields

$$B \in \begin{pmatrix} K^0 & K^2 & K \\ K & K^0 & K^2 \\ K^2 & K & K^0 \end{pmatrix}.$$

The coefficients reduce to the usual complex numbers only locally for a given gauge choice.

With the above identifications, taking into account that every baryon is a colour singlet, we have that the principal bundle associated with it has the form  $(Q^3)^k$  for a suitable integer  $k$ . In particular, it has charge  $ke$ . This means that the  $U(1)$  principal bundles associated with the baryonic fields have indeed a common root  $Q^3$  which is natural to identify with the principal bundle root of the leptonic fields. Therefore, in this model, if quarks are not taken into account the usual (strong) gauge theory applies.

## 7. Cosmological considerations

In the previous sections we have shown that the quantization of charge is implied by the non-exactness of the electromagnetic field. This result leads to a natural physical consequence: the electromagnetic field may manifest its non-exactness at cosmological scales where the non-trivial topology manifests itself and, moreover, it does not need to be singular as in the Dirac's monopole example. However, the electromagnetic field can be non-exact only if the spacetime manifold has a non-trivial cohomology group  $H_{\text{dR}}^2(M, \mathbb{R})$ . We have shown that in this case the electromagnetic field belongs actually to a non-trivial class of  $H^2(M, \frac{2\pi}{q}\mathbb{Z})$  where  $q$  is the charge quantum (this statement also holds in the weak gauge principle case). Now, a globally hyperbolic spacetime is a product  $M = S \times \mathbb{R}$  where  $t \in \mathbb{R}$  is a time function [5]. If  $x^i$ ,  $i = 1, 2, 3$  are coordinates on  $S$  the metric takes the form

$$ds^2 = \chi^2(t, x) dt^2 - R^2(t, x) h_{ij} dx^i dx^j, \quad (60)$$

where  $\chi$  and  $R$  are suitable functions and  $\det h = 1$ . The expansion scalar of the congruence of timelike curves given by  $u = \frac{1}{\chi} \partial_t$  is  $\theta = 3\partial_u \ln R$ ; hence  $R$  can be interpreted as the scale factor of the Universe. Since  $M = S \times \mathbb{R}$ , the electromagnetic field class is proportional to a non-trivial cohomology class of  $H^2(S, \frac{2\pi}{q}\mathbb{Z})$ . This means that the topological non-trivialness of the electromagnetic field arises from its space components i.e. from the magnetic components. Now, unless  $[F] \in H^2(S, \frac{2\pi}{q}\mathbb{Z})$  is a torsion class [6] (this cannot happen, otherwise  $F$  would be exact, see section 4.2) there is a surface  $\Sigma$  in  $S$  such that  $c_\Sigma = \frac{q}{2\pi} \int_\Sigma [F]$  is a constant different from zero known as the first Chern number relative to  $\Sigma$ . The point is that since it is an integer this number cannot change as the Universe expands. Since the area of  $\Sigma$  expands as  $R^2$ , if  $f$  denotes the intensity of the electromagnetic field as measured by a local inertial observer in  $\Sigma$ , its value scales as  $f \sim 1/R^2$ . Thus the expansion of the Universe implies that the actual value of the magnetic field in the non-trivial class decreases. Using the same argument we see that the local energy density of the field scales as  $1/R^4$  exactly as the incoherent radiation does. This kind of behaviour was present since the beginning of the Universe when its topology acquired a final form (at least according to general relativity). The possibility of measuring today a non-trivial cosmological electromagnetic field is then almost ruled out. Even if present, it would have now a negligible value due to the expansion of the Universe. This conclusion is reinforced in those cosmological scenarios that admit an initial inflation.

## 8. Conclusions

After introducing a weak gauge principle, which requires to consider two charged fields, we have studied its implications for electric charge quantization for generic spacetime topologies. We have shown that this new gauge principle has nontrivial implications if the spacetime has torsion in its second integral cohomology group, but it coincides with the standard one if  $H^1(M, U(1))$  (and hence the torsion) is trivial.

If the spacetime has torsion, we have shown that there exist topologically non-trivial configurations of charged fields which do not imply charge quantization. This possibility has not been previously recognized in the literature, although it is compatible with present experimental knowledge. On the other hand, we have proved that charges are quantized on any spacetime whenever the electromagnetic field is not exact.

The weak gauge principle has been exploited, revealing on spaces with torsion a richer structure than the one which would follow from the ordinary gauge principle. In particular we have pointed out that neutral particles can be affected by a topological Aharonov–Bohm

effect, and therefore the interference in such a kind of experiments does not depend solely on the charge of the particle considered but on a topological invariant which is given by a flat line bundle. The comparison of two different particles has revealed the role of the interference class  $k \in H^1(M, U(1))$ . When this class is trivial the phenomenology reduces to that of ordinary gauge theory.

We have shown that the weak gauge principle implies that the  $U(1)$ -bundles determined by the matter fields are not, in general, associated with a common principal bundle. Indeed, they are expressible as a suitable power of a certain non-unique root bundle times a torsion class. This torsion class plays the role of a new quantum number. A torsion subgroup splits the space of  $U(1)$ -bundles  $H^2(M, \mathbb{Z})$  into orbits. Each orbit is identified with a particle while its elements are identified with the different quantum numbers of that particle. We have provided an example considering the case of quarks, where the torsion subgroup is the cyclic group  $\mathbb{Z}/3$  and the quantum number generated is the colour of the particle. We have shown that the weak case is more appropriate in order to describe the quarks and the particle generated from them since the theory naturally embodies two fundamental electric charges, the basic quark charge  $q$  and the electric charge  $3q$ .

We have also suggested that a weak non-exact component of the electromagnetic field over cosmological scales could be responsible for the quantization of the electric charge. Indeed, due to the constancy of the Chern numbers characterizing the non-trivial principle bundle, the expansion of the Universe would make this non-trivial components negligible to the observational capabilities of present day observers. In order to work, this mechanism needs a non-contractible topology of the spacetime manifold, although the non-trivial topology may manifest itself only at cosmological scales. This interesting possibility has the advantage of not being ruled out by observations although it has the related disadvantage of being difficult to verify experimentally.

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